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Difference equations with delays depending on time

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Abstract. For linear difference equations with coefficients and delays varying in time, sufficient conditions are given, in the scalar case, the zero solution to be stable.

1. Introduction

An important class of differential equations of neutral type, are the equations:

$$\frac{d}{dt}Dx_t = f(t, x_t) \tag{1.1}$$

where D is a linear difference operator.

In studying stability and asymptotic behavior of the solutions of (1.1), the major difficulty is related to the properties of the difference operator associated

$$Dx_t = x_t(0) - \sum_{k=1}^{N} A_k x_t(-r_k) \tag{1.2}$$

The importance of the difference operator associated to the differential equations of neutral type, was initially studied by Cruz and Hale [4] and Hale [6].

In the last years, Cruz and Hale [4] Henry [8], Silkowskii [10], Melvin [9], Tsen [11]; Avellar and Hale [1], have been studying stability and asymptotic behavior of solutions of difference equation of the type

$$x(t) = \sum_{k=1}^{N} A_k x(t - r_k) \tag{1.3}$$

with $x \in \mathbb{R}^n, A_k$ an $n \times n$ matrix, and $0 < r_k \leq r$, for $k = 1, \ldots, N$.

In the applications, the coefficients A_k and the delays r_k are not 1 known and we must allow them to vary in some set.

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In the scalar case, Melvin [9] showed that a necessary and sufficient condition for stability of (1.3) is $\sum_{k=1}^{N} |A_k| < 1, A_k \in \mathbb{R}$. Silkowskii [10] extended this result to the matrix case.

In this paper, we take the coefficients A_k and the delays r_k in (1.3) varying in time, that is, we consider the system:

$$\begin{aligned} x(t) &= \sum_{k=1}^{N} A_k(t)x(t - r_k(t)); \quad t \geq 0 \\ x(t) &= \psi(t); \quad -r \leq t \leq 0 \end{aligned} \tag{1.4}$$

and by using a result of Banaś, Hajnosz and Wedrychowicz [2], we prove the following:

Theorem A. *For the system* (1.4), *let's consider:* $r_k\colon [0, +\infty) \to [0, +\infty)$; $A_k\colon [0, +\infty) \to \mathbb{R}$: *real continuous functions,* $A_k(0) = A_k$, $r_k(0) = r_k$, $0 \leq r_k \leq r$; $r_k(t) \leq t + r_k$, $t \geq 0$, $k = 1, \ldots, N$, *and let*

$$r(t) = \begin{cases} \max\{r_k(t), k = 1, \ldots, N\}, & t \geq 0; \\ r = \max\{r_k, k = 1, \ldots, N\}, & -r \leq t \leq 0; \end{cases}$$

Suppose that:

(i) $\sup\limits_{t \geq 0}\{\sum_{k=1}^{N} |A_k(t)|\} < 1$;

(ii) $\lim\limits_{t \to +\infty} (t - r(t)) = +\infty$.

Then, for every $\psi \in C([-r, 0], \mathbb{R}), \psi(0) = \sum_{k=1}^{N} A_k \psi(-r_k)$, *system* (1.4) *has a unique solution* y, *with* $y(t) \to 0$ *as* $t \to +\infty$.

We also give an example showing that in $\mathbb{R}^2$, the theorem is not valid even for the case of periodic delays.

2. Some notations and a fundamental theorem

Let $(E, \| \ \|)$ be a given Banach space. Let us use the following notation:

m_E is the family of all non-empty and bounded subsets of E;

R_E is the family of all non-empty and relatively compact subsets of E.

Let $p(t)$ be a given function defined and continuous on $[-r, +\infty)$ with real positive values, $r \in \mathbb{R}$.

Denote by $C_\rho = C([-r, +\infty); \ \rho(t))$ the set of all real continuous functions defined on $[-r,+\infty)$, such that $\sup\{|x(t)|\rho(t)\colon\ t \geq -r\} < +\infty$.

C_ρ is a real Banach space with respect to the norm

$$\|x\| = \sup\{|x(t)|\rho(t)\colon t \geq -r\}.$$

Next, for an arbitrary $x \in C_\rho, X \in m_{C_\rho}, T > 0, \varepsilon > 0$, let us denote:

$$\omega^T(x,\varepsilon) = \sup\{|x(t)\rho(t) - x(s)\rho(s)|\colon t,s \in [-r,T],\quad |t-s| \leq \varepsilon\}$$
$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon)\colon\ \ x \in X\}$$
$$\omega_o^T(X) = \lim_{\varepsilon\to 0} \omega^T(X,\varepsilon)$$
$$\omega_o(X) = \lim_{T\to\infty} \omega_o^T(X)$$
$$a(X) = \lim_{T\to\infty} \sup_{x\in X}\{\sup\{|x(t)|\rho(t)\colon\ \ t \geq T\}\}$$
$$b(X) = \lim_{T\to\infty} \sup_{x\in X}\{sup[|x(t)\rho(t) - x(s)\rho(s)|\colon\ \ t,s \geq T]\}$$
$$\mu_o(X) = \omega_o(X) + a(X)$$
$$\mu(X) = \omega_o(X) + b(X)$$

The functions $\mu_o(X)$ and $\mu(X)$ are the sublinear measures of non-compactness in the space C_ρ, see [3].

The theorem we will state now is a modified version of the Darbo Fixed Point Theorem (see Banaś, Hajnosz and Wedrychowicz [2]), and it is of fundamental importance to the proof of our Theorem A.

Theorem 1. *Let C be a non-empty, bounded, convex and closed subset of a Banach space E. Suppose $T\colon C \to C$ is a μ-contraction. Then, T has at least one fixed point in C and the set* $\operatorname{Fix} T = \{x \in C\colon\ Tx = x\}$ *belongs to* $\operatorname{Ker}\mu$.

We will also state a result due to Silkowskii [10] related to the stability of the zero solution of equation (1.3).

Theorem 2. *A necessary and sufficient condition for the zero solution of (1.3) to be stable globally in the delays is that:*

$$\gamma_0(A) = \sup\{\gamma(\sum_{k=1}^{N} A_k e^{i\theta_k})\colon\ \theta_k \in [0,2\pi],\quad k = 1,\ldots,N\} < 1$$

where $\gamma(B)$ is the spectral of the matrix B.

3. Proof of Theorem A. Let

$$M = \{y \in C_H([-r,+\infty),\ \mathbb{R}):\ y(t) = \psi(t),\ \ -r \le t \le 0\}$$

where

$$C_H([-r,+\infty),\mathbb{R}) = \{y \in C([-r,+\infty),\mathbb{R}):\ \ \|y\| \le H\}$$

and $F\colon M \to M$ is the map given by:

$$\begin{aligned}(Fy)(t) &= \sum_{k=1}^{N} a_k(t) y(t - r_k(t));\quad t \ge 0\\ (Fy)(t) &= y(t);\quad -r \le t \le 0\end{aligned}$$

It is easy to check that F is a contraction so, by the Banach Fixed Point Theorem, the system (1.4) has a unique solution. To prove stability, we will use the techniques used by Banaś, Hajnosz and Wedrychowicz [2].

First, we make the change of variables:

$$y(t) = x(t)\exp(t - r(t)).$$

So,

$$y(t - r_k(t)) = x(t - r_k(t))\exp(t - r_k(t) - r(t - r_k(t))),$$

and the system (1.4) is equivalent to

$$\begin{aligned}x(t) &= \sum_{k=1}^{N} a_k(t)\exp[r(t) - r_k(t) - r(t - r_k(t))]x(t - r_k(t));\ t \ge 0\\ x(t) &= \exp(r(0) - t)\psi(t);\ -r \le t \le 0\end{aligned} \tag{3.1}$$

Let $C_p = C([-r,+\infty), p(t))$ with $p(t) = \exp(t - r(t))$ and

$$\widetilde{M} = \{x \in C_p\colon x(t) = \exp(r(0) - t)\psi(t); -r \le t \le 0\},$$

and let F be the map defined in $\widetilde{M}$ by:

$$\begin{aligned}(Fx)(t) &= \sum_{k=1}^{N} a_k(t)\exp[r(t) - r_k(t) - r(t - r_k(t))]x(t - r_k(t)); t \ge 0\\ (Fx)(t) &= x(t); -r \le t \le 0\end{aligned}$$

For all $x \in \widetilde{M}$ and $t \geq 0$, we have

$$\begin{aligned}
&|(Fx)(t)|\exp(t-r(t)) = \\
&= \left|\left(\sum_{k=1}^{N} a_k(t)\exp[r(t)-r_k(t)-r(t-r_k(t))]x(t-r_k(t))\exp(t-r(t))\right)\right| = \\
&= \left|\sum_{k=1}^{N} a_k(t)\exp[t-r_k(t)-r(t-r_k(t))]x(t-r_k(t))\right| \leq \\
&\leq \sup_{t\geq 0}\{\sum_{k=1}^{N}|a_k(t)|\}||x||
\end{aligned}$$

As $\sup_{t\geq 0}\{\sum_{k=1}^{N}|a_k(t)|\} < 1$ and $(Fx)(t) = x(t); -r \leq t \leq 0$, we have:

$$||Fx|| \leq ||x||, \forall x \in \widetilde{M}$$

So, F maps $\widetilde{M}$ into itself and transforms the ball $K = K(0,\delta)$ into itself, $\forall\delta \in (0,+\infty)$.

In a similar way, we obtain

$$||Fx - Fy|| < ||x - y||; \quad \forall x, y \in K. \tag{3.2}$$

Therefore, F is continuous in K.

Now, for $X \subset K$, fixed, $x \in X$, $T \geq 0$, $t \geq T$,

$$\begin{aligned}
&|(Fx)(t)|\exp(t-r(t)) = \\
&= \left|\sum_{k=1}^{N} a_k(t)\exp[t-r_k(t)-r(t-r_k(t))]x(t-r_k(t))\right| \leq \\
&\leq \sup_{t\geq 0}\{\sum_{k=1}^{N}|a_k(t)|\}\sup\{|x(t)|\exp(t-r(t)): t \geq \inf_{s\geq T}(s-r(s))\}
\end{aligned}$$

So,

$$a(FX) \leq \sup_{t\geq 0}\{\sum_{k=1}^{N}|a_k(t)|\}a(X) \quad \forall X \subset K, \tag{3.3}$$

fixed.

Now, let us take $\varepsilon > 0, T > 0, t, s \in (0,T)$ such that $|t-s| \leq \varepsilon$ and

$\varphi(t) = t - r_k(t) - r(t - r_k(t)).$

$$|(Fx)(t)\exp(t - r(t)) - (Fx)(s)\exp(s - r(s))| =$$
$$= \left|\sum_{k=1}^{N} a_k(t)\exp(\varphi(t))x(t - r_k(t)) - \sum_{k=1}^{N} a_k(s)\exp(\varphi(s))x(s - r_k(s))\right| \leq$$
$$\leq \sup_{t\geq 0}\{\sum_{k=1}^{N} |a_k(t)|\} \sup\{|A(t) - B(s)|:\ t, s \in (0,T),\ |t - s| \leq \tau\}$$

where

$$A(t) = \exp(t - r(t))x(t);$$
$$B(s) = \exp(s - r(s))x(s), \quad \text{and}$$
$$\tau = \sup\{|(t - r(t)) - (s - r(s))|, t, s \in (0,T), |t - s| \leq \varepsilon\}.$$

So,

$$w_o^T(FX) \leq \sup_{t\geq 0}\{\sum_{k=1}^{N} |a_k(t)|\} w_o^T(X), \quad \forall X \subset K, \tag{3.4}$$

fixed.

Combining (3.3) and (3.4), we obtain:

$$\mu_o(FX) \leq \sup_{t\geq 0}\{\sum_{k=1}^{N} |a_k(t)|\}\mu_o(X), \quad \forall X \subset K,$$

fixed, which means that F is a μ_o-contraction, and by Theorem 1, the system (3.1) has a fixed point x in C_p, and $\lim_{t\to+\infty} x(t)p(t) = 0$.

But, $x(t) = \exp[-(t - r(t))]y(t)$, and so $\lim_{t\to+\infty} y(t) = 0$.

4. Matrix case

Consider the system

$$\begin{aligned} x(t) &= A_1 x(t - r_1) + A_2 x(t - r_2); t \geq 0 \\ x(t) &= \varphi(t); -a \leq t \leq 0 \end{aligned} \tag{4.1}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$; $A_1 = \begin{bmatrix} \frac{2}{3} & 0 \\ 0 & -\frac{2}{3} \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & \frac{2}{3} \\ -\frac{2}{3} & 0 \end{bmatrix}$, $\alpha \in \mathbb{R}$, $r_1 > 0$, $r_2 > 0$, $a = \max\{r_1, r_2\}$.

It is easily checked that, the zero solution of (4.1) is stable globally in the delays according by Theorem 2.

Consider now, $r_1 = r_1(t)$, $r_2 = r_2(t)$, continuous periodic functions with period 3, such that:

$$r_1(0) = 1, \quad r_1(1) = 3, \quad r_1(2) = 3,$$
$$r_2(0) = 2, \quad r_2(1) = 1, \quad r_2(2) = 1,$$

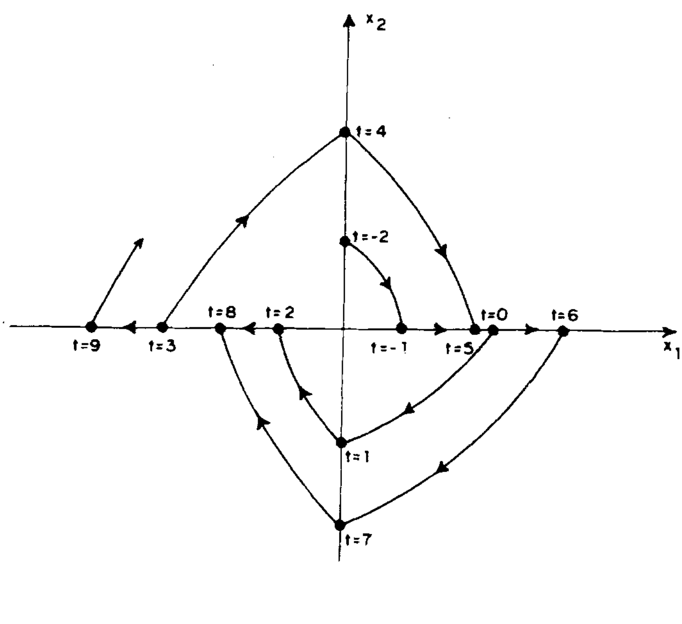
Let $\varphi: [-2, 0] \to R^2$, with $\varphi(-1) = e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\varphi(-2) = e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and consider the system:

$$\begin{aligned} x(t) &= A_1 x(t - r_1(t)) + A_2 x(t - r_2(t)); \quad t \geq 0 \\ x(t) &= \varphi(t); \quad t \in [-2, 0] \end{aligned} \tag{4.2}$$

Now, observe that:

$$\begin{aligned} x(-1) &= e_1; \\ x(0) &= \frac{4}{3} e_1; \\ x(1) &= -\frac{14}{9} e_2; \\ x(2) &= -\frac{10}{27} e_1; \\ x(3) &= -\frac{104}{81} e_1; \\ x(4) &= \frac{460}{243} e_2. \end{aligned}$$

Geometrically, we have the following situation in the phase space.



By induction, we can to show that, the sequence $\{x(6j)\}_{j=0,1,\ldots}$ is such that $x(6j)\not\to 0$ as $j \to \infty$, and so (4.2) is not stable.

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